Sociétés humaines et survie: un modèle de processus de ramification.

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Résumé

Depuis ses débuts, l'humanité a mis à l'épreuve une pléthore de formes très différentes de sociétés, ce qui pose un certain nombre de questions fort intéressantes. Le but de cet exposé est de présenter un modèle général de population et les réponses qui s'en suivent, sur la base des deux hypothèses naturelles suivantes: d'une part une volonté globale de survie et d'autre part un désir individuel d'un niveau de vie meilleur. Parmi d'autres conclusions non triviales, nous démontrerons notamment que deux sociétés particulières constituent une "quasi-enveloppe" pour toutes les sociétés possibles. Ce résultat, montré rigoureusement par des méthodes probabilistes, est, à notre connaissance, nouveau. Par ceci nous arrivons aussi à une conclusion, qui nous semble remarquable: étant donné que l'une des sociétés "limites" peut être considérée comme une société proche d'un communisme extrême, et l'autre comme une version d'un capitalisme pur et dur, nous voyons que l'humanité n'est plus loin d'avoir testé ses limites.

Mots clés: Processus de Galton-Watson, processus de branchement contrôlés, critères d'extinction, convergence presque-sûre, convergence complète, statistiques d'ordre, temps d'arrêt, courbe de Lorenz, sociétés arbitraires, mercantilism, communism, capitalism.

AMS classification de sujets: 69J85, 60J05;
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Human societies and survival:
A Branching Process Model

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Abstract

From its early beginnings onwards mankind has put to test many different society forms, and this fact raises a complex of interesting questions. The objective of this lecture is to present a general population model which takes the essential features of any society into account and which can give answers on the basis of two most natural hypotheses: on the one hand a clear intention to survive, and on the other hand the desire to increase the standard of living. Among several non-trivial conclusions we show that two society forms stand out in the sense that they form a quasi-envelope of all possible societies. As far as we aware, this result, obtained in a probability model and proved rigorously, is new. Among these results we also arrive at a conclusion which strikes us as remarkable: given that one of the "limiting" societies is close to an extreme form of communism, and the other one similarly close to an extreme version of capitalism, we see that humanity is not far away from having already experienced its limits.

Keywords: Galton-Watson processes, controlled branching processes, extinction criteria, almost-sure convergence, complete covergence, order statistics, expected stopping times, Lorenz curve, arbitrary society structures, mercantilism, communism, capitalism.

AMS subject classification: primary 69J85, 60J05;
secondary classification 60G40.

1 Introduction

What is it that keeps societies together? What is the goal of any society? Are there natural boundaries for societies mankind would not or cannot exceed? And if so, can we identify and quantify parameters characterizing these boundaries?

The first question is rather philosophical, and we only treat it in as much as it concerns the subsequent questions. Here we shall provide a mathematical answer obtained from a model we propose as a global mathematical model for societies. This model is built on branching processes and submitted to two natural hypotheses. Still rudimentary, the model is broad enough to cope with essential features of life within any society: reproduction of individuals, the desire to have a future, heritage and production of resources, consumption of resources, individual preferences and competition, policies to distribute resources among individuals, and, as a tool of interaction, the right of emigration. We look at different sub-models of the model, characterizing different societies. These are defined by the type of control they exercise through different policies to distribute resources among their individuals.

1.1 Objectives of societies

Any society is likely to advertise certain keywords in its program or mission statement, such as justice, liberty, equal opportunity, security, etc. We all agree that these issues are likely to be important. However, for several of these issues there may be as many different interpretations as there are individuals in a population. Hence, within a whole population they can hardly serve as reliable guidelines for the choice of a specific society form.

The philosophy of our approach to answering questions about the choice of a society is therefore to focus on factors which are seen as dominant, namely those which come out of two natural and inoffensive hypotheses:

> **Hypothesis 1**: Individuals want to survive and to see a future for their descendants.
>
> **Hypothesis 2**: Individuals prefer a higher standard of living to a lower one.

Since these hypotheses may not always be compatible with each other we define Hypothesis 1 to have a higher priority than Hypothesis 2. Other hypotheses may be implicit, as for instance the desire to have security. It is implicit in Hypothesis 2. If the standard of living is sufficiently high, the society can afford to provide protection through a qualified police force and a strong army.

To deal with these hypotheses in an adequate way, the problem is to find a suitable model. This requires three important conditions. First, the model should allow for all those features which are seen as essential for the development of a human society; second, it should permit interaction of individuals and society. And third, the model should be sufficiently tractable to allow for quantifiable conclusions.

1.2 History of results

The speaker has been thinking about ways to model societies for many years and had given a talk on *resource dependent branching processes* as early as in 1983.

At a first step only preliminary results about necessary conditions for survival were obtained, and this only for an elementary model. These results were based on earlier work on branching processes with random absorbing processes and on the so-called counterpart of the Borel-Cantelli-Lemma (B. (1980)).

In a second step, several models were tested. When seeing, in a different context, the article by Coffman, Flatto and Weber (1987), their results were sharpened to our needs in B. and Robertson (1991). These results implied real progress in the direction of quantifiable conclusions and still play the same important role in today's lecture. Depending on the specific model under investigation, they made it possible to find explicit survival criteria. However, at that time the model versions were still immature in the sense that they were not convincing as models for a human society.

In a third phase, the speaker realized that whenever he incorporated more realistic features in a model, two society forms stood out. On the one hand, this was what one could call the *strongest-first-society* (sf-society), and, on the other hand, the *weakest-first-society* (wf-society). The question was then whether the critical parameters for survival of *any* society would lie in between the respective critical parameters of these two special societies. The outcome was a first form of *a* theorem of *envelopment* presented in a talk at Ecares, ULB, (B. 2001). However, the speaker was less sure whether this was the best possible result and did not submit it for publication.

The question of a best possible result for the chosen model took time, and profited from new attempts through trial and error. It profited also from stimulating discussions with Mitia Duerinckx, ULB, of which the outcome is the article B. & Duerinckx (2012) (submitted for publication). This article gives a precise formulation of *the* Theorem of envelopment of societies, and its proof. This Theorem says that all possible societies are bound to live *in the long run* between two extreme societies. In fact, these extreme societies are the weakest-first society and the strongest-first society mentioned before.

The present lecture will explain the ideas of the proofs behind all the results and discuss several well-known society forms under the light of these results. Moreover, the lecture will show that the envelopment of societies, combined with the quantitative results obtained before, is, however simple it may look, a fundamental result.

2 The model

We consider a population, beginning at time 0 with a fixed number of individuals, which reproduce at distinct times $n = 1, 2, \cdots$. The time interval $[n, n+1[$ is called the nth *generation.* Individuals consume resources and create new resources for their descendants. Only those descendants whose resource claims will be met by the society will stay within the population until the next reproduction time, the others are supposed to emigrate (or die) before reproduction.

We first define all the components of the model. In order to keep notations simple we use indices without brackets or other forms of distinction. In the rare instances where we mean to say "power" instead of upper index, this is clearly indicated.

2.1 Reproduction

Each individual is supposed to reproduce independently of each other. The model supposes that reproduction is asexual. The number of descendants of each individual is modeled according to a common probability law $\{p_k\}_{k=1,2,\dots}$, where p_k denotes the probability that a given individual will have exactly k descendants (children). To avoid trivial cases we suppose $p_0 > 0$ and a few other natural conditions specified below. Let D_k^j denote the number of descendants of the jth individual in the kth generation. Hence

$$P(D_k^j = \ell) = p_\ell,\ j = 1, 2, \cdots; k = 1, 2, \cdots; \ell = 0, 1, 2 \cdots . \tag{1}$$

Asexual reproduction is assumed for simplicity. Passing to bisexual reproduction and so-called mating functions would make the precise description of a population process in several cases much more complicated but here the essence of the results would not change as we will show later.

The infinite double-array $(D_j^k)_{j=1,2,\dots}^{k=1,2,\dots}$, named *reproduction matrix*, thus consists of independent identically distributed (iid) random variables, and their mean is denoted by $m := \mathrm{E}(D_j^k)$.

2.2 Resources

Human beings need food, they need resources. They also reproduce, and thus they need resources for their descendants. Hence they must save resources and create resources for future generations. Resources are distributed in each generation, the planning is for the longer-time future development of the population, however.

In our model individuals inherit resources from preceding generations. They consume resources and create new resources. The resources an individual can use during its lifetime determines its standard of living, and we define it to be its standard of living. The society decides in what way resources are distributed among the individuals, or expressed differently, it is the acceptance of policies to distribute resources what defines a society. The inherited resources, plus the newly created ones, are, after deduction of consumption, considered as being the individual contribution to society.

We do not distinguish for individuals between heritage and new production of resources, and neither between different types of production or consumption of resources. The term resources is to be understood in a very large sense ranging from goods and materials up to education and knowledge. Everything is summarized as individual *creation* of resources and seen as the individual's contribution to a *common* resource space available for the next generation. Resource creations of individuals are modeled as iid random variables $R_j^k, j = 1, 2, \dots; k = 1, 2, \dots$, and the infinite double array

$$\left(R_j^k\right)_{j=1,2,\dots}^{k=1,2,\dots} \tag{2}$$

will be called *resource creation matrix*. We suppose that $r := \mathrm{E}(R_j^k) < \infty$.

2.3 Objective of survival

The population's desire to survive is understood as the objective to have for the society as a whole a positive probability of surviving *forever.* If certain rules to distribute resources allow for a positive probability of survival whereas other rules would not achieve this, then the objective to survive is taking priority, and the rules would be changed accordingly. It suffices here to see this as an omnipresent option and to think of the rules defining the society, even if they had been changed many times before, as being fixed from today onwards for the whole future.

2.4 Resource claims within a society

The model interprets each individual claim of resources of an arbitrarily chosen descendant as being the outcome of two random components. One is the descendant's desire to have a certain amount of resources, the other one being what this descendant, with its own power of conviction, will be able to defend among its competitors within the society.

The random claims of individuals are modeled as independent random variables governed by a known *continuous* distribution function

$$F(x) = P(X_j^k \leq x);\ \mu := \mathrm{E}(X) < \infty. \tag{3}$$

If there are t_n descendants in the nth generation then they generate a string of claims

$$(X_1^n, X_2^n, \cdots, X_{t_n}^n), \tag{4}$$

called the *string of claims* in the nth generation.

The idea is of course that the latter may be chosen as a function of the distribution of resource production. Again, for our objective in the present paper, we consider both distributions as fixed for the future, and the choice (or persuasion) of a claim distribution as the society's tool of control.

2.5 Interaction of individuals and Society

Further we need a control instrument for individuals. Each individual is supposed to have the right to emigrate, and the control instrument is the right to exercise this option. To fix the rules, we suppose that an individual emigrates or dies if and only if its individual resource claim is not *completely* satisfied; otherwise it remains a member of the population until the end of the generation. Emigration or death are supposed to happen before an individual produces descendants. Hence each individual resource assignment, seen as the standard of living offered by the society, is felt by an individual as being either sufficient, implying *stay*, or else insufficient, implying *leave.*

Policies: Typically, the total resources inherited or produced by a generation are insufficient to satisfy all the resource claims of the descendants. We define a policy as a function which determines a priority order among the current string of claims, that is, a rule to distribute the resources created by the current generation among their descendants.

2.6 Natural regularity conditions

To avoid trivial cases we suppose that the following natural assumptions are always satisfied: (i) $m > 1$, because otherwise no survival is possible. Indeed, there is no immigration, and the reproduction of the individuals is independent like in a Galton-Watson process. (ii) For a human society we must insist on $p_0 > 0$. Hence from (i) we require $k \geq 2$ with $p_k > 0$ for some $k \geq 2$. (iii) The trio of laws of reproduction, creation of resources, and claims should be compatible with a positive probability, however small it might be, that the process can reach any finite state, because otherwise it is bound to die out.

In addition, we suppose that all random variables have *finite variances.* This has a technical reason and does not qualify as a regularity condition, but still as a natural condition. Indeed, even the stronger assumption that all variables are bounded can be defended.

3 Resource Dependent Branching Processes

We now give a precise definition of the type of population processes we consider in this study. Two definitions are needed.

Definition 3.1: A *policy* π in generation n with t_n descendants and available resource space s_n is a function π, which assigns *priorities* to the string of claims $(X_n^1, X_n^2, \cdots, X_n^{t_n})$, that is a permutation $(X_n^{i_1}, X_n^{i_2}, \cdots, X_n^{i_{t_n}})$ with the rule to serve $X_n^{i_1}$ before $X_n^{i_2}$ before .... before $X_n^{i_{t_n}}$ as long as the currently available resource space s_n suffices.

The relevant outcome of the policy is then the number of individuals which stay until reproduction, that is the value of the counting function

$$Q^\pi(t_n, (X_n^k)_{k=1}^{t_n}, s_n) = \max\{0 \leq j \leq t_n : \sum_{k=1}^{j} X_n^{i_k} \leq s_n.\} \tag{5}$$

Definition 3.2: A *Resource Depending Branching Process* (*RDBP)* with policy π on (D_n^k, X_n^k, R_n^k) is the counting process $(\Gamma_n)_{n\in\mathbb{N}}$, defined through

$$\Gamma_0 = 1,$$

and recursively

$$\Gamma_{n+1} = Q^\pi\left(D_n(\Gamma_n), (X_n^k)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)\right), \ n = 1, 2, \cdots,$$

where $D_n(\Gamma_n)$ is short notation for the total number of descendants of Γ_n, and, correspondingly $R_n(\Gamma_n)$ for the total ressource space created by Γ_n.

Remark: Each process in this lecture is supposed to start at time 0 at level 1 unless another initial value is clearly indicated.

3.1 Examples of policies.

To keep examples simple we use here positive integers for claims and available resources; this is not required in reality, of course. Any random claim expresses the number of units of resources the individual insists to receive; otherwise it will emigrate or die. Assume for instance that, in a given generation, the number of individuals is 10, and that the current resource space is 100. Suppose further that the individuals write down their claims on a list in some order, as for instance in alphabetic order of names, and that the string of claims reads

$$11, 7, 15, 19, 11, 18, 10, 22, 17, 19.$$

We write "1" if the society agrees to grant this claim, and "0" otherwise. Then

$$1\ ,1\ ,1\ ,\ 1\ ,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,$$

would present the "alphabetic priority" society. It grants the first seven claims (adding up to 91), and the last three applicants would emigrate. (The remaining units, 9 in this case, may be split among those who can stay, say, but such details do not matter in the model as we will see).

A society that distributes resources in alphabetic order has not much appeal, as one may argue, but there are certainly more foolish ways, as e.g. the "coin-flipping society". Such a society assigns at each claim at random a "0" or "1", and , if the total of resources were not used up, would it start anew with the remaining ones as long as there are resources left.

Indeed, any procedure to select a priority of claims is considered as a policy, and any policy can be seen as an rule to assign 0's and 1's to the vector of claims.

Two policies will attract our special interest in the following Section: One satisfies the smallest claims first, which, in the very first example above, retains the claims $7, 10, 11, 11, 15, 17, 18$. This is the wf-policy (weakest-first), and the society applying this policy is the wf-society. The other special society, the sf-society (strongest-first) retains the largest claims first, that is, in our example, $22, 19, 19, 18, 17$.

4 Weakest-first-society and strongest-first-society

Since the last two society form play a special role throughout, we give their precise definition.

4.1 wf-society

Definition 4.1 The *weakest-first policy* π^W serves smallest claims first with counting function

$$N(t, (x_k)_{k=1}^t, s) = \begin{cases} 0, & \text{if } t = 0 \text{ or } x_{1,t} > s, \\ \sup\left\{1 \le k \le t : \sum_{j=1}^k x_{j,t} \le s\right\}, & \text{otherwise,} \end{cases} \tag{6}$$

where $x_{1,t} \leq x_{2,t} \leq \cdots \leq x_{t,t}$ denote the *increasing order statistics* of the t iid random variables $(x_k)_{k=1}^t$, and s the available resource space.
The *weakest-first process* on $(D_n^k, X_n^k, R_n^k)_{n,k}$ is defined by $W_0 = 1$, and recursively,

$$W_{n+1} = N\left(D_n(W_n), (X_n^k)_{k=1}^{D_n(W_n)}, R_n(W_n)\right), n = 1, 2, \cdots . \tag{7}$$

0 • ...|........|............|........................|..................................|.... • s

Figure 1

Figure 1 shows an example where starting by serving the smallest claims first, 5 claims are met completely; thus 5 individuals stay in the population to generate the next generation.

Lemma 4.1: The counting function $N(t, (x_k)_{k=1}^t, s)$ is increasing in both t and s with the other variable being held fixed.

4.2 sf-society

Definition 4.2 The *strongest-first*-policy is defined through the counting function

$$M(t, (x_k)_{k=1}^t, s) = \begin{cases} 0, & \text{if } t = 0 \text{ or } x_{t,t} > s \\ \sup\left\{1 \leq k \leq t : \sum_{j=t-k+1}^t x_{j,t} \leq s\right\}, & \text{otherwise,} \end{cases} \tag{8}$$

where, as before, the $x_{1,t} \leq x_{2,t} \leq \cdots \leq x_{t,t}$ denote the increasing order statistics of the t iid random claims.
Correpondingly, the *strongest-first process* on $(D_n^k, X_n^k, R_n^k)_{n,k}$ is defined by $S_0 = 1$, and recursively,

$$S_{n+1} = M\left(D_n(W_n), (X_n^k)_{k=1}^{D_n(S_n)}, R_n(S_n)\right), n = 1, 2, \cdots . \tag{9}$$

Figure 2

0 • ....|....................................|.... • s

This figure shows an example where starting by serving the largest claims first, only 2 individuals can stay in the population to generate by reproduction the next generation.

We can also show the following Lemma:

Lemma 4.2 The counting function $M(t, (x_k)_{k=1}^t, s)$ is *increasing* in the variable s with the variable t being fixed and *unimodal* in t with s being held fixed.

If resources are plenty and suffice to accommodate all claims, then both policies have the same effect, i.e., they allow all individuals to stay and reproduce. However, if not, the wf-society is the one which allows the maximum number of individuals to stay, and to reproduce. The sf-society is then opposite in the sense that the resource space is used up by the corresponding minimum number of applicants. In fact, it is easy to see from the Definitions 4.1 and 4.2:

Corollary 4.1 For any RDBP (Γ) with arbitrary counting function Q^Γ we have

$$M(t,(x_k)_{k=1}^t,s) \le Q^\Gamma(t,(x_k)_{k=1}^t,s) \le N(t,(x_k)_{k=1}^t,s). \tag{10}$$

However, as we shall see in the next Section, the sf-society is not opposite to the wf-society in every aspect.

5 Main results

1. **Markov property.** RDBPs are Markov processes with a unique absorbing state, which is 0.

2. **Uniform upper bound.** Let (Γ_n) be an arbitrary RDBP. Then it follows from (6) and (7) that

$$\Gamma_n \le W_n \text{ almost surely for all } n = 1, 2, \cdots . \tag{11}$$

Hence the wf-process (W_n) dominates almost surely and uniformly in time any arbitrary RDBP. (Note that this inequality holds for the whole processes and is much stronger that the rhs-inequality of Corollary 4.1.)

3. **"Safe-haven" property of the wf-process.** Let

$$q_\Gamma = P(\lim_{n\to\infty} \Gamma_n = 0 | \Gamma_0 = 1)$$

be the probability of final extinction of the process (Γ), and let q_W be the final extinction probability for the wf-process. Then

$$q_W \le q_\Gamma. \tag{12}$$

$$P(\lim_{n\to\infty} W_n = 0 | W_0 = j) \le q_W^j, \tag{13}$$

where the superscript j means here "power j." The first inequality is immediate from (11). The second one, somewhat harder to show, has a remarkable implication: If $q_W < 1$, and this is the only case of interest, then q_W^j becomes quickly very small as the size j increases. Hence any society which is not too small in size, is very likely to avoid distinction by switching to the wf-policy. The wf-society may therefore be seen as the "safe-haven" society with respect to Hypothesis I.

4. **Non-existence of a uniform lower bound.** In general we do *not* have $S_n \leq \Gamma_n$ a.s. for all n. An explicit counterexample can be constructed based on Lemma 4.2. and, more precisely, on the fact that $M(t, (x_k)_{k=1}^t, s)$ is, for fixed s, increasing in t up to some threshold t_s but decreasing for $t \geq t_s$.

This means that (S_n) is not a complete opposite of (W_n), and the phenomenon can be made intuitive as follows: Suppose that $Q^S \neq Q^\Gamma$, and that, for some generation n, these two societies happen to have equal size, i.e. $S_n = \Gamma_n = L$, say.

Then, since the distribution of the number of descendants and the distribution of their resource claims is identical for both societies, Γ_{n+1} is likely to be larger than S_{n+1} because S retains in general less individuals from the descendants of the L individuals than Γ. It follows that the number of descendants of Γ_{n+1} is then also likely to be larger than the number of descendants of S_{n+1}. This again implies that the string of resource claims of the descendants of Γ_n is likely to become longer than the string of the descendants of S_n.

However, the longer string is more likely to contain large claims than the shorter one. Hence, if Q^Γ accommodates at least one large claim not present in the shorter string of claims of the descendants of S_{n+1}, then the inequality can change into the other direction, that is $S_{n+2} > \Gamma_{n+2}$.

5. **Theorem of the envelopment of societies.**

The non-existence of a uniform lower bound makes several conclusions harder. The more surprising is that, nevertheless, we can prove the following strong result:

Let $(S_n(j)), (\Gamma_n(j))$ and $(W_n(j))$ denote respectively, the sf-process, an arbitrary RDBP, and the wf-process, each starting at time 0 with initial size j. Note that this generalizes the definition of $(S_n), (\Gamma_n)$ and (W_n) by setting $S_n := S_n(1), \Gamma_n := \Gamma_n(1)$ and $W_n := W_n(1)$. Then we have

Theorem 5.1

$$\text{(i) } P\left(\lim_n \Gamma_n(j) \leq \lim_n W_n(j)\right) = 1, \quad \text{for all } j = 1, 2, \cdots . \tag{14}$$

Moreover, for all $\epsilon > 0$ there exists a sufficiently large population size j_0 such that

$$\text{(ii) } P\left(\lim_n S_n(j) \leq \lim_n \Gamma_n(j) \leq \lim_n W_n(j)\right) > 1 - \epsilon, \text{ for all } \ j \geq j_0. \tag{15}$$

The proof of Part (i) follows easily from (10). We do not prove Part (ii) here but point out that the major result used in this proof is Theorem 2.2 of B. and Robertson (1991) the application of which is shown below for the proof of the equally important results of Subsection 6.

6. **Critical curves for survival.**

Recall that the average reproduction rate per individual is $m = \mathrm{E}(D_k^j)$, the average resource creation $r = \mathrm{E}(R_k^j)$, and the average resource claim $\mu = \mathrm{E}(X_k^j)$ with all rates being expressed per individual.

We say that the mean resource creation rate r is *critical* with respect to a fixed mean reproduction m (average number of descendants) and a fixed resource claim distribution F if a strictly lower rate $r' < r$ leads to almost sure extinction whereas a strictly larger rate $r'' > r$ allows for a strictly positive survival probability. The results are as follows:

Suppose that $r/m \leq \mu$. Let τ and ϑ be the solutions of

$$\int_0^\tau x dF(x) = \frac{r}{m} \text{ and } \int_\vartheta^\infty x dF(x) = \frac{r}{m}. \tag{16}$$

Then the critical curves for possible survival are as follows. For the wf-society we have

$$\text{critical wf-society:} \qquad mF(\tau) = 1 \tag{17}$$

For the sf-society we have in the case of bounded resource claims

$$\text{critical sf-society:} \quad m(1 - F(\vartheta)) = 1. \tag{18}$$

We also note that we may consider in each generation the total available resource space as being bounded. Hence the constraint of bounded resource claims for the result for the sf-society is a most reasonable assumption. .

6.1 Example. As a simple example, suppose that resource claims are iid uniform random variables on $[0, u], u > 0$. We compute the critical value for the wf-process. We have $F(x) = x/u$ on $[0, u]$ so that

$$\int_0^\tau x dF(x) = \int_0^\tau x(1/u)dx = \frac{\tau^2}{2u} = \frac{r}{m}.$$

Hence $\tau = \sqrt{2ru/m}$ so that $mF(\tau) = 1$ implies the critical value $r_c(\text{wf}) = u/(2m)$. Similarly, the critical value for the sf-society is seen to be $r_c(\text{sf}) = u(2m-1)/(2m)$. This is $(2m-1)$times as much as the critical value for the wf-process $r_c(\text{wf})$. Thus, if $m = 2$ for instance, individuals living in the wf-society on the critical value of creation, and wanting to change to the sf-society, must increase their average resource creation by factor 3 to be able to survive in the long run; indeed, an enormous difference.

6.2 Main steps of proofs.

Let us first prove the result $mF(\tau) = 1$ for the critical curve of the wf-process:

It follows from the Markov property of RDBP's with single absorbing state 0 that, in the long run, only extinction or final explosion are possible. Also, no RDBP can survive if it is throughout bounded in size. This is shown in more generality for branching processes with arbitrary absorbing processes in (B.1978) for which the conditions are satisfied in the case of RDBP's. Indeed, since $p_0 > 0$ the only absorbing state in a bounded process would then always be accessible with a probability bounded away from 0.

The natural regularity condition (iii) assures that any RDBP can reach any size with a positive probability, however small this probability might be.

According to the Borel-Glivenko Theorem, as the sample size increases, the empirical distribution function of resource claim sizes converges uniformly to the real distribution function F. Moreover, since all random variables D_j^k in the reproduction matrix, all X_j^k in the claims matrix, and all R_j^k in the resource creation matrix are iid and have finite second moments, the arithmetic means taken row-wise in each matrix, i.e. within the same generation, over the respective strings, all converge completely. See Hsu and Robbins (1948), Theorem 1, page 26. (See also Asmussen and Kurtz (1980) for extensions.)

But this implies from Theorem 3. on page 30 of Hsu and Robbins (1948) that these arithmetic means in each double-array of random variables, taken over increasing generations, converge almost surely, and this means to the respective means m, μ, and r. Since convergence of any sequence to its limit implies convergence of any subsequence to the same limit we obtain almost sure convergence of all arithmetic means of the relevant strings in each of the three double-arrays.

Now Theorem 2.2 and Theorem 2.3 of B. and Robertson (1991) result become the central tools. We show it here for the first one.

Let $X_1, X_2, \cdots$ be iid positive random variables with continuous distribution function F. Let $(\Phi_n)_n \to \infty$ and $(\Psi_n) \to \infty$ such that $\Psi_n/\Phi_n \to \rho$ almost surely with $0 < \rho \le \mathrm{E}(X)$. Further let τ be the solution of

$$\int_0^\tau x \mathrm{d}F(x) = \rho \tag{19}$$

Then we have

$$\frac{1}{\Phi_n} N\left(\Phi_n, (X_k)_{k=1}^{\Phi_n}, \Psi_n\right) \to F(\tau) \text{ almost surely as } n \to \infty. \tag{20}$$

Now let $\Phi_n = D_n(W_n) = \#\{\text{descendants of } W_n\}$ and $\Psi_n = R_n(W_n)$ be the random total resource space created by W_n. Then

$$\Psi_n/\Phi_n \to \rho := \frac{r}{m} \text{ almost surely as } W_n \to \infty. \tag{21}$$

From (7) we obtain

$$\frac{1}{D_n(W_n)} N\left(D_n(W_n), (X_n^j)_{j=1}^{D_n(W_n)}, R_n(W_n)\right) = \frac{W_{n+1}}{D_n(W_n)}, \tag{22}$$

which tends, according to (15), to $F(\tau)$ almost surely as $W_n \to \infty$. But this implies from (18) and (20) that

$$\frac{W_{n+1}}{W_n} \to mF(\tau) \text{ almost surely as } W_n \to \infty. \tag{23}$$

Now, if $mF(\tau) > 1$ then there exists a real value $\tilde{m}$ such that $1 < \tilde{m} < mF(\tau)$. It follows from almost sure convergence of the arithmetic means of strings that, for all $\epsilon > 0$ there exists a sufficiently large level L_ϵ such that

$$P\left(\bigcap_{\ell=1}^{\infty}\left\{\frac{W_{n+\ell}}{W_{n+\ell-1}} > \tilde{m}\,\middle|\, W_n \geq L_\epsilon\right\}\right) > 1-\epsilon. \tag{24}$$

Here it suffices to show, that the process (W_n), once being above the level L_ϵ will only fall at most finitely often below this level. This implies then that, if $mF(\tau) > 1$, the process (W_n) will attain with a strictly positive probability a level from which onwards it will finally grow at least as quickly as an exponentially growing process with rate $\tilde{m} > 1$, and thus avoid extinction.

Moreover, we can show by similar arguments that if $mF(\tau) < 1$ then the process (W_n) will be bounded in expectation. From this one can show that its extinction probability must be one. Hence $mF(\tau) = 1$ is the critical value for survival of the wf-process, and (17) is proved.

The proof of (18) is quite similar. However, here we must use the assumption that the resource claims are bounded as assumed in the model. This was not needed for the proof of (17) where we only used that the claims have a finite variance.

This proves the critical curves for the wf-society and the sf-society.

We now return to Part (ii) of Theorem 5.1.

Look at the asymptotic behavior of Γ_{n+1}/Γ_n for an arbitrary RDPB with counting function Q^Γ. We have

$$\frac{\Gamma_{n+1}}{\Gamma_n} = \frac{1}{\Gamma_n} Q^\Gamma\left(D_n(\Gamma_n), (x_k^n)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)\right) \tag{25}$$

$$\geq \frac{1}{\Gamma_n} M\left(D_n(\Gamma_n), (x_k^n)_{k=1}^{D_n(\Gamma_n)}, R_n(\Gamma_n)\right) \tag{26}$$

where the inequality follows from (10) in Corollary 4.1. If $\Gamma_n \to \infty$ then, as shown before, $D(\Gamma_n)/\Gamma_n \to m$ almost surely and $R(\Gamma_n)/\Gamma_n \to r$ almost surely.

Now, provided that $m(1 - F(\vartheta)) > 1$ (see (18)) the last ratio must finally at least exceed any real value m^* with $1 < m^* < m(1 - F(\vartheta))$. Hence, if $m(1 - F(\vartheta)) > 1$ then we can show with similar arguments as before that the process (Γ_n) has a strictly positive probability to finally dominate a process with exponential growing rate greater than one, and thus a strictly positive probability to avoid extinction. Since both (S_n) and (Γ_n) have only the two possible limits 0 and ∞, this yields (15). □

5.1 Significance of the results

The fact that the survival criteria for both extreme societies can be given explicitly makes the Envelopment Theorem significant. Note again that these theorems give extinction/survival criteria in terms of the parameters m (mean offspring number), r (mean resource creation), and the distribution function of resource claims F (from which we also know the mean resource claim μ.) In each case the solution of a last relevant parameter (τ and ϑ, respectively) is obtained by solving a simple integral equation. Thus the critical boundaries are explicit. (We note that, interestingly, this equation involves the *Lorenz curve* which is well-known in Economics.)

Now recall the "safe-haven" property.

We have seen that if the survival probability of the wf-society is strictly positive, then, however small it may be when starting with few individuals, it converges quickly to 1 with increasing size. And hence we concluded that, provided $q_W < 1$, any society has always the option of a very probable survival by letting converge their rules, if necessary, towards the rules of the wf-society. If $q_W = 1$, however, then, with a fixed offspring probability law $\{p_k\}$, society must draw the consequences, because, viewing the chance of survival, there is no alternative. The individuals live beyond their means and must be instructed by the society to either become more modest in average claims of resources or else to increase the average reproduction of resources.

No other society in this model does as much for ensuring survival as the wf-society. The price to pay under the same fixed distribution is the most modest standard of living of individuals in this society.

The sf-society forms the other extreme. Under the given assumptions this society does the most for the standard of living of the few. However, it jeopardizes the prospects of survival more than any other society.

Both *extreme* societies form an *quasi-envelope* for any society in the sense that, in the long run, no society can exceed these bounds. We call it a quasi-envelope because the wf-society leads to a definite, uniform upper bound process whereas the sf-society leads only to a very probable lower bound process.

However, one can also show that there cannot be a better definite lower bound process for all processes. It lies in the nature of the problem and in our intention to cover all possible policies that no further improvement is possible.

5.2 Tractability of the model

Clearly, RDBP's are still relatively simple models compared with what we expect we would need to model societies in a most realistic way. However, there are strong reasons why they should earn our attention.

Firstly, of course, it is not realistic to look out for a perfect model, and, keeping this in mind, RDBP's seem to be a good approach because they give considerable room for modeling aspects.

Secondly, RDBP's yield, as we have seen, not only a strong Envelopment Theorem for societies but also explicit survival criteria in form of quantifiable critical relationships between society forms. This fact should not be taken for granted. As we have seen in the general definition of RDBP's, realistic society forms will typically impose complicated structures. Almost all interesting forms are too complicated to be tackled by generating functions or martingale arguments, the most powerful tools in branching process theory.

Thirdly, RDBP's are remarkably robust. The main results flowing from them hold in more general settings. So, in particular, consider the assumption that reproduction within a RDBP is asexual (Galton-Watson process assumption). It is interesting to know what happens if we replace this assumption by the natural assumption that reproduction depends on two sexes. (See Molina (2010) for a review of known results in this domain.)

The answer is in fact in favor of RDBP's. Since survival is only possible if the RDBP can grow without limits, the asexual reproduction mean m can here be substituted by the so-called limiting *average reproduction mean.* The average reproduction mean for a total of k mating units $m(k)$ is defined in equation (1) of B.(1984). In the notation of the present paper it translates into

$$m(k) = \frac{1}{k}\mathrm{E}(D_n(\tilde{\Gamma}_n)\Big|\tilde{\Gamma}_n = k), \tag{27}$$

where, unlike Γ_n, the modified $\tilde{\Gamma}_n$ counts now he number of "mating units" (and not individuals) present in generation n, and $D_n(\tilde{\Gamma}_n)$ denotes the number of mating units generated by these for the next generation. If $\ell := \lim_{k\to\infty} m(k)$ exists - and this is the case for the majority of natural mating functions - then the specific form of the mating function becomes irrelevant for extinction criteria as soon as the population size has become sufficiently large. (See also Daley et al. (1986).) Hence the generalization to sexual reproduction may affect the initial chances of reaching larger numbers of individuals within a RDBP but does not affect the main results.

Similarly, a little reflection shows that passing form discrete time generations to more realistic "moving" generations makes it technically harder to define the precise meaning of strings of resource claims. However, under some reasonable conditions there are ways around the formal problems and moving generations do not impair the set-up, and although the speaker has not gone though all details, critical relationships between the society form and the parameters $\tilde{m}, r, \mu$ can seemingly be found accordingly.

5.3 Real-World Conclusions

Returning to the RDBP's we have defined, we will comment for the remainder of this lecture on real-world conclusions by confining our interest to the important ones.

On the one hand we have some intuition that all societies we may think of should have, for fixed probability laws of natality, of resource production, and of resource consumption, somewhere their limits. On the other hand, as we have seen, this intuition is partially wrong and requires a thorough revision. Rigorous arguments then helped to overcome the new difficulties. These arguments lead to more subtle conclusions. The more remarkable is that, after refinement, a major part of the original intuition stays still true.

It is tempting to apply these results by looking at society forms we see around us, or of which we know that mankind has tried them before. Much insight may be gained from learning why certain society forms have failed, and why others seem to do, or to have done, relatively well. It would be nice to see that scientists who have access to data or estimates needed for the analysis presented here will find such questions a real challenge.

5.3.1 Theory and reality

One often hears the adage "theory and reality are two completely different things." What is typically meant by this is that the hypotheses in a theory are sometimes not satisfied, or rarely completely satisfied, so that the conclusions may not hold. However, the simplified and frequently advertised adage "What works in theory does not work in practice" is a nonsense simplification. If it were true then the equivalent proposition "what works in practice does not work in theory" would be true, but the latter is clearly false.

Hence, the only remaining meaningful version of the adage is "what does not work in theory cannot work in practice", and this is thus the interpretation which will guide us through our short comparison of our results with real-life societies. Since we characterized societies by policies to satisfy individual resource claims and regarded the distribution functions of reproduction, of production of resources, etc, as being fixed, such comparisons are limited. Nevertheless, they are of interest.

6 A short comparison of major society forms.

In the following we shortly discuss the main features of a few selected societies, and how they can be seen as RDBP's.

6.1 Mercantilism.

Mercantilism was the dominant policy for western societies for most of the 16th century up to the end of the 18th century, and in some countries even to the beginning of the 19th century. There are several forms of mercantilism, and it is not easy to unify answers. Nevertheless, with respect to RDBP's, there is a common denominator to all different forms of mercantilism, that is, as we shall argue, a state-controlled "head-and-tail-policy" of distributing resources.

The philosophy of mercantilism is that the wealth of a nation, compared with the wealth of other nations, is a zero-sum game. This implies that leaders concentrate their interest on the competition between different states for a common fixed wealth of the world. At the same time the idea behind was also that a rich country can afford a strong army to defend wealth. Strict mercantilists, exemplified by Colbert in France, concluded that all what counts is getting the wealth into the own country by exports and keeping production costs of goods as low as possible. A positive balance of trades was the main concern; imports were highly taxed.

In the interpretation of RDBP's, the members of the government or kingdom as well as the rich merchants, are typically those with the large claims. They form the "head", i.e. the largest claims, but they are relatively small in numbers (see Figure 3).

The "tail" consists of the claims of those individuals in the population who have to produce (farmers, workers, ...) In agreement with the philosophy of mercantilism that money which is spent is also lost, production cost was kept as low as possible. The tail is made up the modest claims, but now to be satisfied form a reduced resource space.

$$0 \bullet \text{ largest claims...} \rightarrow \text{.....} \bullet \bullet \text{ weakest first}|.|.|.|.|.|..|..| \uparrow \bullet s$$

Figure 3

Figure 3 displays the essence of mecantilism with respect to the distribution of claims. Relatively few claims take a major part of the resource space, whereas the other claims resemble wf-society claims on the head-reduced resource space.

The right of emigration (the control in our model), was denied in some countries during certain periods, and in such cases unsatisfied claims must be interpreted as removal by death.

As we understand today, mercantilism suffered from a lack of experience, or, as harsh critics would say, a lack of understanding. The rules of import and export were very strict, and the idea that free trade creates value and that wealth is by far not a zero-sum game had to await the arrival of great economists, like, among others, Adam Smith.

6.2 Enlightened mercantilism.

Although economists like Dudley North, John Locke, and Adam Smith in particular undermined much of mercantilism, and saw themselves, or were called by others, anti-mercantilists, one may call them today also *enlightened* mercantilists. The new ideas of the great value of free trade and of a motivated work force seemed, almost revolutionary, and it is true that they brought a great change and opened the way to more recent economic societies. However, at the beginning a part of the philosophy of mercantilism was still in force. So for instance the priority of inexpensive production before fair compensation for work was still rather present. Also, not all defendants of the new ideas accepted already the idea that the wealth of nations is no zero-sum game.

The graph of claims in a society under an enlightened mercantilism would therefore still have some resemblance with that of Figure 3. However there are two clear differences: Nations became richer once the barriers to free trade imposed by the classical philosophy of mercantilism were relaxed. Hence the available resource space s should be seen as getting significantly larger. Also, the barrier between small claims and large claims became less clear-cut because the number of people involved in and profiting from trade grew steadily.

All forms of mercantilism are of course strictly inside the envelope, but the earlier forms of this society form implied some typical disadvantages of both the wf-society and the sf-society to which we return below.

6.3 Modern western societies

As much as we might see large differences between the nations in western Europe as for instance Belgium, France, Germany, Great Britain, etc, we are constantly reminded that everything is just relative. Seen as RDBP's with their graph of resource claims, these societies are not that different.

Figure 4

$0 \bullet$ |.....|.....|.....|.....|.....|.....| $\cdots$ |......|......| $\cdots \rightarrow$ larger claims $\uparrow \bullet s$

Figure 4 tries to display the typical resource claims picture of western societies. If we order the claims in increasing order, lower claims begin above a lower bound. They are similar in size in a large part of the population. The upper part is very mixed.

These societies, indeed most western societies, including Australia, New Zealand, and many others, including states from the former eastern block, have, on the one hand a rather solid social security system. On the other hand, they display society forms ranging from a (controlled) free society up to a laissez-faire society. The first feature implies that the smallest claims are, compared to the larger ones, much bigger than in mercantilism societies, and the second one that the larger claims go through a wide range. (In US-society low and large claims seem both to go through larger ranges).

6.4 Capitalism

As indicated before, the strongest-first society shares features of an extreme form of capitalism. The large claims stem from the class of people who have the power to defend them within society and are thus strongly correlated with financial power. The graphics of the resource claim distribution would look similar to the one for the sf-society (see Figure 2).

In one point, things are of course different. It suffices to ask who would effectively still create the resources which are made available by manual work. Thus, even in extreme forms of capitalism, there must still be some space for the lower claims.

6.5 Communism

Correspondingly, the weakest-first society shares undeniably features with an extreme form of communism. If means of production are completely in the hands of the people's society the larger claims are no longer backed by personal power due to individual ownership. Hence the smaller claims, backed by the masses, are automatically given priority.

Again exception must be made for the top political class but in comparison to the whole population the number of individuals in this class is very small. Hence the overall picture of the resource claim distribution over the whole space of currently available resources would have much similarity with the one of the wf-society (see Figure 1.)

6.6 Extreme communism versus extreme captitalism

If we accept the comparison of the wf-society and the sf-society with extreme versions of communism and capitalism, respectively, then, according to our results, these two societies form the quasi-envelope of all society forms.

It is a strange envelope indeed, namely, its boundaries attract and repulse individuals at the same time. Neither of the extreme societies can be expected to be seen by individuals as attractive, at least not if resources are limited, and this is typically the case.

People want to get away from extreme communism by Hypothesis 2. They would like to increase their standard of living as far as possible and thus leave extreme communism as quickly as possible.

In the extreme form of capitalism (sf-society), things are equally unstable. Neglecting the case of arbitrarily many resources for everybody, the greed of the strongest ones kills the population from above because typically a large proportion of people will leave. This society will soon have to revise thoroughly its philosophy. One reason is, as shown in this article, the necessity to have more people to increase the survival probability. Another reason is that there must be enough people to make the present resources available.

It is difficult to imagine that extreme capitalism could possibly survive in reality. In our RDBP model it is possible under the condition $m(1 - F(\vartheta)) > 1$. However, this condition imposes not only a high productivity but is also the result of a model in which individual resource creations stay iid random variables. Work forces from different society classes are in reality not always easily exchangeable.

6.6.1 Marxism.

Returning to Hypothesis 2 and our conclusion that extreme communism will never last long we give, in a certain way, reason to Marx for the facts, although not for the conclusion. Indeed, Marx saw that it may be hard to convince people of the advantages of his new ideology. We may ask whether it was worth convincing people to adapt a society form to which they will not want to be faithful, but having said this, one must try to give a balanced view.

Marx saw things differently. He believed that the "Mehrwert " (added value) of the labor of the working class would then mainly benefit the working class who *creates* value, and that, as a result, income of the lowest class would go up. In RDBP's this does not happen unless the mean production per individual would go up. Again, one has to give to Marx that he was hoping this would also be the case, i.e. that not only the income of the lower class but also the overall productivity would go up. Many opponents of communism are convinced of the very contrary, namely that the loss of personal advantages in the ideal communistic society will bring productivity down.

Some opponents say that this lies in the nature of human beings, others argue that statistical evidence suffices to show this.

We have to stay neutral on the latter discussion because our analysis touches only the critical relationship between demand and productivity, and not what is behind. What RDBP's show (compare Example 6.1) is that the added value would have to be very high to convince people that their search for a higher standard of living would be of little importance.

Also, we must stay neutral with comments on what we see today as catchy slogans or provocative arguments in communist ideology.

For example, when Marx formulated his famous promise "From each according to his ability, to each according to his needs", then there is a priori nothing remarkable about this in a RDBP. If a low productivity of an individual comes together with a large claim of the same individual then this must be expected to happen in a model with iid random variables. Any society can propose to serve claims independently of their size, provided that the resource space allows for it. But this is the point. To know whether the resource space will allow for the advertized generosity, Marx would face again the crucial question of productivity.

6.6.2 Leninism

Coming back to the question of added value, Lenin was, compared to Marx, much more radical in his views. He went so far as advocating the new ideas to be imposed by a "vanguard party approach", i.e. a revolution. And this is what he got. Lenin and his followers cannot be made responsible for all the unpredictable deviations and sad excesses which followed in the politically established dictatorship after him, but some serious questions should be asked here.

How come Lenin dared to push towards a revolution before knowing more about the alleged added value by moving production tools in the hands and property of the working class? What does not work in theory cannot work in practice, so how could Lenin know it would work in theory? This was completely new territory, there was no prior experience, there were no data. It is difficult to imagine that Lenin could possibly know more about the added value than Marx himself?

Was is only for personal ambitions of Lenin to seize power? All the speaker can say is that he has not found a convincing argument which would justify Lenin's radical approach to impose a revolution with such enormous risks. Specialists may know better.

6.6.3 Comparing ideologies

As we have seen, when maintaining the mean resource creation constant, extreme communism cannot be a stable society. It has one *undeniable and distinguished*advantage worth repeating: With fixed resource creation (productivity) it is superior to any other society with respect to survival probability, whereas extreme capitalism has no comparable distinguished advantage.

However, individuals wish to increase their standard of living and Hypothesis 2 will push extreme communism more and more into consumption, i.e. into a "stronger"-first society of some kind. If this move stays without control then it will approach capitalism, the declared enemy of communism.

Capitalistic arguments that the prosperity of the few will always benefit those who have to live with the bare minimum cannot be refuted through our model with a fixed resource production. However, it finds no more support than the added-value arguments of Marx.

No wonder capitalism and communism are declared enemies. There is more to it than just a great difference of ideology. Seen as RDBP's the difference between them is fundamental because they are as opposed to each other with respect to the natural Hypoetheses 1 and 2 as they possibly can be. The extreme communism is doing everything for Hypothesis 1, and nothing for Hypothesis 2, and the extreme capitalism does exactly the opposite. However, both opponents are born losers, and there is little reason for born losers to be declared ennemies. Societies may test these limits, as they have done, but any attractive society is bound to be somewhere else within the envelope or quasi-evelope of societies.

6.7 A modest outlook.

It is not ideology that really changes the world, but ideas. Consequently, it is tempting to comment on great economists like Smith, Keynes, Samuelson and many others. It is equally tempting to say something about different economies in the world, or even point-wise, about economic projects like the "great deal" of Roosevelt in the US, or the Agenda 2000 of Schröder in Germany. Our results cannot offer much for such questions because here our models are too rudimentary.

If mankind has seen several other variations of societies which seem more attractive than the extreme societies we had to discuss before because they stood out, then this is due to an increase of understanding. Understanding the mechanisms and the impact of financial instruments of a national economy, including all the instruments of monetary policy and labor policy, these are essential factors as we believe today, and there is little reason to doubt it.

One thing is clear, and will always be true: If the model approach via RDBP's is accepted, then the interesting move of any society will always be towards something in the middle of the envelope, and this envelope is formed by the wf-society and the sf-society. Not accepting this conclusion means not agreeing with the Hypotheses 1 and 2. As far as the speaker is aware, nothing as definite as this has ever been stated and proved before.

Finally, could data help mankind to converge in its search for an optimal policy towards an "optimum"? Hopefully yes towards something like an optimum region. Nevertheless, as we have seen before, Hypotheses 1 and 2 will always be in force. Fluctuations of policies, even around an alleged optimum society believes to have found, will therefore always be part of the game.

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